

Problem Set 5

September 27, 2008

1. Compute the rank of the following matrix: $\begin{bmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{bmatrix}$

2. Evaluate the determinants.

(a) $\begin{bmatrix} 1 & 3 & 2 \\ 8 & 4 & 0 \\ 2 & 1 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & a & 1 \\ 1 & 0 & b \\ c & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$

3. Determine the values of t for which the matrix

$$\begin{bmatrix} t & 1 & 0 \\ 0 & t & 0 \\ 0 & 0 & t+3 \end{bmatrix}$$

is invertible.

4. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be invertible matrices of the same order. Simplify the expressions

(a) $(I + A)A^{-1}(I - A)$

(b) $A(3A^{-1} + 4B^{-1})B$

(c) $(AB^{-1}C)^{-1}$

5. Use Cramer's Rule to solve the following equation systems:

(a)

$$\begin{aligned}3x_1 - 2x_2 &= 11 \\2x_1 + x_2 &= 12\end{aligned}$$

(b)

$$\begin{aligned}10x + y - 2z &= 30 \\x - y &= 1 \\3x - 4y + z &= 15\end{aligned}$$

6. Find the ranks of the following matrices .

(a) $\begin{bmatrix} 3 & 0 \\ 5 & 1 \\ 9 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 9 & 11 \end{bmatrix}$

7. Let $\mathbf{Ax} = \mathbf{b}$ be a linear system of equation in matrix form. Prove that if \mathbf{x}_1 and \mathbf{x}_2 are both solutions of the system, then so is $\lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2$ for every real number λ . Use this fact to prove that a linear system of equations that is consistent has either one solution or infinitely many solutions. (For instance, it cannot have exactly 3 solutions.)

8. An orthogonal matrix is an invertible square matrix whose inverse is its transpose.

9. Show that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

are orthogonal matrices.

10. Prove that the determinant of an orthogonal matrix is either 1 or -1 . Show that both cases can occur.

11. Let a, b, c be constants such that $abc \neq 1$. Use Cramer's rule to solve the system of equations

$$\begin{aligned}ay + z &= 1 \\x + bz &= 1 \\cx + y &= 1.\end{aligned}$$

12. Write the following quadratic form in matrix form: $5x_1^2 - 10x_1x_2 - x_2^2$

13. Determine the values of t for which the symmetric matrix $\begin{bmatrix} 2t & 2 \\ 2 & t \end{bmatrix}$ is

- (a) positive definite
- (b) positive semidefinite but not positive definite
- (c) negative definite
- (d) negative semidefinite but not negative definite
- (e) none of the above

14. Determine the definiteness of the symmetric matrices

(a)
$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 9 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 6 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

15. Let $\mathbf{A} = (a_{ij})_{n \times n}$ be symmetric and positive semidefinite. Prove that \mathbf{A} is positive definite if and only if $|\mathbf{A}| \neq 0$.