Problem Set 4

September 19, 2008

1. For the CES production function

$$Q = A\left[\delta K^{\gamma} + (1-\delta) L^{\gamma}\right],\ A > 0, \gamma < 1, 0 < \delta < 1$$

derive the second-order conditions for unconstrained maximzation with two choice variables

2. Let $y = f(x_1, x_2)$ and let $z = F(y) = F(f(x_1, x_2)) = g(x_1, x_2)$. Show that if $F' > 0$, then g has a stationary point at (x_1^0, x_2^0) when and only when f is stationary there. Under what conditions will f have a maximum when and only when g has a maximum. What are the implications of this?

3. Find the stationary point and determine whether it is a relative maximum, minimum, or saddle point of $f(x_1, x_2)$

   (a) $f(x_1, x_2) = -4x_1 - 6x_2 + x_1^2 - x_1x_2 + 2x_2^2$
   (b) $f(x_1, x_2) = 12x_1 - 4x_2 - 2x_1^2 + 2x_1x_2 - x_2^2$

4. A firm uses two inputs to produce a single product. If its production function is $Q = x^{\frac{1}{4}}y^{\frac{1}{4}}$ and if it sells its output for a dollar a unit and buys each input for \$4 a unit, find its profit-maximizing input bundle.

5. Dingbat airlines has regular flights between Ypsilanti and Kalamazoo.

   (a) It can treat business and pleasure travelers as separate markets by demanding advance purchase and Saturday night stay-over for pleasure travelers. Suppose that it notes a demand function of $Q = 16 - P$ for business travelers and a demand function $Q = 10 - P$ for pleasure travelers and that it has a cost function for all travelers of $C(Q) = 10 + Q^2$. How much should it charge in each market to maximize its profit?

   (b) Compute the demand function for the market as a whole, without price discrimination. Compute the firm's profit maximizing output for this situation and compare the profit to profit earned in the previous problem.

6. Show that the rate of change of output with respect to a factor price change is equal to the negative of the rate of change of that factor with respect to output price, (ie, eq $(4-27)$ in Silberberg).

7. Consider the production function $y = x_1^{\alpha_1} x_2^{\alpha_2}$. Find the factor demand curves and the comparative statics of a profit-maximizing firm with this production function. Show that $\frac{\partial x_2^*}{\partial w_1} < 0$.

8. Consider the following two definitions. "Factor 1 is a substitute (complement) for factor 2 if the marginal product of factor 1 $x_1$decreases (increases) as factor 2 is increased". "Factor 1 is a substitute (complement) for factor 2 if the quantity of factor 1 employed increases when the price of factor 2 increases (decreases)"

   (a) Show that these two definitions are equivalent in the two-factor, profit maximization model.

   (b) Do you think these definitions will be equivalent in a model with 3 or more factors? Why?

9. Consider Ex 3, Sec 4-2 (Silberberg) where a monopolist sells his or her output in two separate markets. Suppose a per-unit tax is placed on output sold in the first market.

   (a) Show that an increase in t will reduce the output sold in market 1.

   (b) Show that it is possible that an increase in the tax on market 1 can lead to an increase in total output $x^*(t) = x_1^*(t) + x_2^*(t)$, even assuming the usual sufficient second-order conditions. Under what circumstances (slopes of the MC and MR functions) does this occur? (This possibility is know as the *Hotelling taxation paradox* after Harold Hotelling, who first explored it.)

10. Consider a profit-maximizing firm with the production function $y = f(x_1, x_2)$, facing output price p and factor prices w_1 and w_2. Suppose this firm is taxed according to the total cost of factor 2, ie, $tax = tw_2x_2$.

    (a) Derive the factor demand functions. Are they homogeneous of any degree in any of the factors?

    (b) Show that if the tax rate rises, the firm will use less of factor 2.

    (c) Show that $\frac{\partial x_1^*}{\partial t} = w_2 \frac{\partial x_2^*}{\partial w_1}$

    (d) Suppose that factor 1 is held fixed at its profit-maximizing level. Show that the response of factor 2 to a change in the tax rate is less in absolute value than before.

11. Consider a profit-maximizing firm with production function $y = f(x_1, x_2)$ that sells its output competitively at price p. The firm obtains input x_1 at a competitively determined unite wage w_1 but the firm faces an upward-sloping supply function of x_2 given by $w_2 = w_2^0 + kx_2$ where w_1, w_2^0, p, and k are positive parameters.

    (a) Derive the first and sufficient second order conditions. Is the "law of diminishing marginal product" implied for each factor.

    (b) Derive the comparative static results available for the parameter w_1. What refutable implications are forthcoming, if any?