Proof Strategies*

Doing proofs is analgous to doing puzzles: you sort of fool around with it, do part of the border (the beginning and the end), do some other easy chunks, and then try to figure out how they connect. Similar idea with a proof.

Typically in economics, you will encounter *theorems*. A theorem usually states some *hypotheses* (assumptions) and says that if the hypotheses hold, then some conclusion must hold.

In order for a theorem to be correct it must be the case that whenever the hypotheses are correct, the conclusion is also true. Note that this was the same criteria we used when doing truth tables for judging when a statement was valid. It turns out that much of doing proofs is simply re-writing the problem into an easier form. The equivalencies we established while doing truth tables will prove very useful in rewriting proofs.

In your econ courses, you will encounter a large number of theorems. Let's look at a few so that you can see the form:

Example 1 *If the binary relation $\succeq$ is complete, transitive, continuous, and strictly monotonic, there exists a continuous real-valued function, $u : R_+^n \to R$, which represents $\succeq$.*

Example 2 *Let $\succeq$ be a preference relation on R_+^n and suppose $u(\mathbf{x})$ is a utility function that represents it. Then $v(\mathbf{x})$ also represent $\succeq$ if and only if $v(\mathbf{x}) = f(u(\mathbf{x}))$ for every $\mathbf{x}$, where $f : R \to R$ is strictly increasing on the set of values taken on by u.*

Example 3 *Let preferences $\succeq$ over gambles in G satisfy axioms $G1$ to $G6$. Then there exists a function $u : G \to R$ representing $\succeq$ on G, such that u has the expected utility property.*

If you can find a counterexample to a theorem then the theorem must be incorrect; however, just because you find an example of when the theorem is true does not mean that the theorem is correct. In order to be correct, the theorem must be hold for all possible examples. We must prove the theorem. We will outline a number of strategies for solving proofs.

Remark 4 ***To prove $P \to Q$, assume P is true and then prove Q***

Recall our truth tables. We know that this strategy works because we showed that when P is true and Q is true then $P \to Q$ is also true.

Typically when doing a proof, you will have make some transformation in the proof and then prove the transformation. For that reason, lets adopt the following notation when discussing proofs.

*More from **?**

We will call statements that are assumed or figured out during the course of the proof *givens*. We will call the statement that you are trying to prove the *goal*. When you start a proof, your givens will be the hypotheses of the theorem you are trying to prove; as you work on the proof, the givens will include other statements that you infer from the hypotheses or other assumptions that you make. The goal is initially the conclusion of the theorem but it can also change several times during the proof.

We'll first look at an illustrative example.

Example 5 *Suppose a and b are real numbers. Prove that if $0 < a < b$ then $a^2 < b^2$.*

Remember you typically have to fool around with a proof so lets show the scratch work that we would use solve it.

Our strategy tells us to assume P is true. So,

Staring at this we that $0 < a < b$ is pretty close to $a^2 < b^2$; why not square $0 < a < b$. We get $0 < a^2 < b^2$ which is what we wanted to show.

Theorem 6 *Suppose a and b are real numbers. Prove that if $0 < a < b$ then $a^2 < b^2$.*

Proof. Suppose $0 < a < b$. Squaring this we get $0 < a^2 < b^2$ as required. Thus, if $0 < a < b$ then $a^2 < b^2$. ■

Notice the basic form the proof took:

- Suppose P
  - [Proof of Q goes here]
- Therefore, $P \to Q$

Remark 7 ***To prove $P \to Q$, assume Q is false and prove P is false***

This strategy isn't surprising. Recall that we showed using truth tables that the contrapositive $\neg Q \to \neg P$ is equivalent to $P \to Q$. So the basic form of this proofing method will be:

- Suppose $\neg Q$

– [Proof of $\neg P$ goes here]

- Therefore, $P \to Q$

Example 8 *Suppose a, b, and c are real numbers and $a > b$. Prove that if $ac \leq bc$ then $c \leq 0$.*

The contrapositive of the goal is $\neg(c \leq 0) \to \neg(ac \leq bc)$ or $(c > 0) \to (ac > bc)$. So,

Multiply $a > b$ by c. This gives us $ac > bc$.

Theorem 9 *Suppose a, b, and c are real numbers and $a > b$. Prove that if $ac \leq bc$ then $c \leq 0$.*

Proof. We will prove the contrapositive. Suppose $c > 0$. Multiplying both sides of the given inequality $a > b$ by c we conclude $ac > bc$. Therefore, if $ac \leq bc$ then $c \leq 0$. ∎

Note: when writing the final form of your proof you should English rather than symbolic notation.

It often isn't clear which proofing strategy to use. You need to try several and see which works.

1 Proofs involving negations and conditionals

Remark 10 ***To prove a goal of the form $\neg P$, rewrite and use previous strategies.***

Usually it is easier to prove a positive statement than a negative one.

Example 11 *Suppose $A \cap C \subseteq B$ and $a \in C$. Prove that $a \notin A \backslash B$.*

The goal is rather complicated. In words, it says that it is not possible that a is an element of A but not an element of B. You might want to sketch it out with a Venn Diagram first to see what it is saying. Lets also rewrite.

$$\begin{aligned} a &\notin A \backslash B = \neg(a \in A \backslash B) \\ &= \neg(a \in A \wedge a \notin B) \\ &= a \notin A \vee a \in B \\ &= a \in A \to a \in B \end{aligned}$$

So,

Lets try strategy 1: assume P, prove Q. Then

Now if $a \in C$ and $a \in A$ then $a \in (A \cap C)$. Since $A \cap C \subseteq B$ we can then conclude $a \in B$.

Theorem 12 *Suppose $A \cap C \subseteq B$ and $a \in C$. Prove that $a \notin A \backslash B$.*

Proof. Suppose $a \in A$. Then since $a \in C$, $a \in (A \cap C)$. But then since $A \cap C \subseteq B$ it follows that $a \in B$. Thus it cannot be the case that a is an element of A but not of B, so $a \notin A \backslash B$. ■

Sometimes you can't express $\neg P$ as a positive statement. Then try to do a proof by contradiction.

Remark 13 ***To prove a goal of the form $\neg P$: Assume P is true and try to reach a contradiction. If you reach a contradiction, then you can conclude that P must be false. (Proof by Contradiction)***

The reason this works is that if P is false, then by definition, $\neg P$ must be true, which is what we originally wanted to show.

- Suppose P
  - [Proof of contradiction goes here]
- Therefore, P is false

Example 14 *Prove that if $x^2 + y = 13$ and $y \neq 4$ then $x \neq 3$.*

- $x^2 + y = 13$ and $y \neq 4$
  - [Proof of $x \neq 3$ goes here]

- Thus, if $x^2 + y = 13$ and $y \neq 4$ then $x \neq 3$

We first think about rewriting $x \neq 3$ as a positive: $\neg(x = 3)$. There are no logical connectives anywhere, so this doesn't get us any further. So we think about doing a proof by contradiction.

- $x^2 + y = 13$ and $y \neq 4$
  - Suppose $x = 3$
    - [Proof by contradiction goes here]
  - Therefore, $x \neq 3$
- Thus, if $x^2 + y = 13$ and $y \neq 4$ then $x \neq 3$

Once we've done all this, the proof is easy. If $x = 3$ then we know that $y = 13 - 9 = 4$. But this is a contradiction. Therefore we can conclude that $x \neq 3$.

Theorem 15 *Prove that if $x^2 + y = 13$ and $y \neq 4$ then $x \neq 3$.*

Proof. Suppose that $x^2 + y = 13$ and $y \neq 4$. Suppose that $x = 3$. Substituting this into the equation $x^2 + y = 13$ we find that $y = 4$. But this is a contradiction. Therefore $x \neq 3$. Thus we conclude that if $x^2 + y = 13$ and $y \neq 4$ then $x \neq 3$. ■

Remark 16 ***The key to solving proofs is often writing out the definition***

To use a given of the form $\neg P$:Typically try to rewrite
To use a given of the form $P \to Q$:

- Since you are given $P \to Q$, if you are given P or can prove that P is true then you can also conclude that Q is true.
- If you can prove Q is false then you can conclude that P is false since $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$

Example 17 *Suppose $P \to (Q \to R)$. Prove that $\neg R \to (P \to \neg Q)$*

Note: we could show this using a truth table but lets practice our proofs! Following our typical strategy,

So the structure of our proof is:

- Suppose $\neg R$
  - Proof of $P \to \neg Q$ goes here
- Therefore, $\neg R \to (P \to \neg Q)$.

Following the same logic,

So, the structure of our proof is:

- Suppose $\neg R$
  - Suppose P
    - Proof of $\neg Q$ goes here
  - Therefore, $P \to \neg Q$
- Therefore, $\neg R \to (P \to \neg Q)$

Now note that we can apply our tip from above.

- Suppose $\neg R$
  - Suppose P
    - Since P and $P \to Q$, then $Q \to R$
      - Proof of $\neg Q$ goes here

  - Therefore, $P \to \neg Q$

- Therefore, $\neg R \to (P \to \neg Q)$

From given 1 and 3 we can conclude that $Q \to R$ is true. This means that $\neg R \to \neg Q$. Since we know $\neg R$ then we conclude that $\neg Q$ is true.

Theorem 18 *Suppose $P \to (Q \to R)$. Prove that $\neg R \to (P \to \neg Q)$*

Proof. Suppose $\neg R$. Suppose P. Since P and $P \to (Q \to R)$ then $Q \to R$. Since $\neg R$ then $\neg Q$. Thus, $P \to \neg Q$. Therefore, $\neg R \to (P \to \neg Q)$. ■

Exercise 19 *Suppose $A \subseteq B, a \in A$, and a and b are not both elements of B. Prove that $b \notin B$.*

2 Proofs Involving Quantifiers

Theorem 20 *Suppose A, B, and C are sets, $A \backslash B \subseteq C$, and x is anything at all. Prove that if $x \in A \backslash C$ then $x \in B$.*

Note the statement "x can be anything at all". In other words, x can take on any value; it is arbitrary. So in the proof, we in fact want to show that $\forall x \{x \in A \backslash C \to x \in B\}$. More generally, this theorem is asking you to prove something of the form $\forall x \; P(x)$ where $P(x)$ is $x \in A \backslash C \to x \in B$. The basic idea of this type of proof is you need to prove it very generally, so that it holds for all possible x. In other words, you don't want to make any assumptions about x when doing your proof.

Remark 21 *To prove a goal of the form $\forall x P(x)$: Let x stand for an aribtary object and prove $P(x)$.*

The basic form of our proof is:

- Let x be arbitrary.
  - Proof of $P(x)$ goes here
- Since x was arbitrary, we can conclude that $\forall x P(x)$.

First,

So the form of our proof goes:

- Let x be arbitrary
  - Suppose $A \backslash B \subseteq C$
    - Proof of $x \in A \backslash C \to x \in B$ goes here
  - Thus, if $x \in A \backslash C$ then $x \in B$.
- Since x was arbitrary, we can conclude that $\forall x P(x)$.

Let's try strategy 1: assume P, prove Q.

- Let x be arbitrary
  - Suppose $A \backslash B \subseteq C$
    - Suppose $x \in A \backslash C$

      - · [Proof of $x \in B$ goes here]
    - $*$ $x \in B$
  - Thus, if $x \in A \backslash C$ then $x \in B$.

- Since x was arbitrary, we can conclude that $\forall x P(x)$.

Since there are no logical connectives, we can't do much with $x \in B$. So let's try a proof by contradiction.

- Let x be arbitrary
  - Suppose $A \backslash B \subseteq C$
    - $*$ Suppose $x \in A \backslash C$ and $x \notin B$
      - · [Proof by contradiction goes here]
    - $*$ $x \in B$
  - Thus, if $x \in A \backslash C$ then $x \in B$.
- Since x was arbitrary, we can conclude that $\forall x P(x)$.

So lets write out what we know:

$$\begin{aligned} A \backslash B &\subseteq C \text{ means } (x \in A \wedge x \notin B) \rightarrow x \in C \\ x &\in A \backslash C \text{ means } x \in A \wedge x \notin C \end{aligned}$$

We have assumed that $x \notin B$ and that x is anything. If $x \in A$ and $A \backslash B \subseteq C$ we know that $x \in C$. But this is a contradiction. Therefore, we can conclude that $x \in B$.Thus, if $x \in A \backslash C$ then $x \in B$.

Theorem 22 *Suppose* A, B, *and* C *are sets,* $A \backslash B \subseteq C$, *and* x *is anything at all. Prove that if* $x \in A \backslash C$ *then* $x \in B$.

Proof. Let x be arbitary. Suppose $x \in A \backslash C$. This means that $x \in A$ and $x \notin C$. Suppose $x \notin B$. From $A \backslash B \subseteq C$ we know that if $x \in A \wedge x \notin B$ then $x \in C$. But this contradicts the fact that $x \notin C$. Therefore, $x \in B$. Thus, if $x \in A \backslash C$ then $x \in B$. Since x was arbitrary, we can conclude that $\forall x \{x \in A \backslash C \rightarrow x \in B\}$ so $A \backslash C \subseteq B$. ■

Example 23 *Suppose* A *and* B *are sets. Prove that if* $A \cap B = A$ *then* $A \subseteq B$.

Let's tackle the goal and rewrite.

$$A \subseteq B \text{ means } \forall x \, (x \in A \rightarrow x \in B) \, .$$

We can't do anymore fussing with the goal so lets look at our givens. Since $x \in A$ and $A \cap B = A$ then it must be the case that $x \in A \cap B$. This implies that $x \in B$, which is what we wanted to show.

Theorem 24 *Suppose A and B are sets. Prove that if $A \cap B = A$ then $A \subseteq B$.*

Proof. Let x be arbitrary. Suppose $A \cap B$. Suppose $x \in A$. Since $x \in A$ and $A \cap B = A$ then it must be the case that $x \in A \cap B$. Thus, $x \in B$. Since x was arbitrary, we can conclude that $\forall x \, (A \cap B = A \rightarrow x \in B)$ so $A \subseteq B$. ■

Recall that the other type of quantifier was $\exists$. Just as its name suggest, you just need to find one case where the theorem holds, in order to prove it.

Remark 25 ***To prove a goal of the form*** $\exists x P(x)$ *:Try to find a value of x for which you think x will be true. Start your proof with "Let $x =$ (the value you decided on)" and prove $P(x)$ for this value of x.*

Exercise 26 *Prove that for every real number x, if $x > 0$ then there is a real number y such that $y(y+1) = x$.*

3 Proofs involving conjunctions and biconditionals

Remark 27 ***To prove a goal of the form*** $P \wedge Q$ *: Prove* P *and* Q *separately*

Remark 28 ***To use a given of the form*** $P \wedge Q$ *: Treat them as two separate givens*

Exercise 29 *Suppose* $A \subseteq B$ *and* A *and* C *are disjoint. Prove that* $A \subseteq B \backslash C$*.*

Remark 30 ***To prove a goal of the form*** $P \leftrightarrow Q$ *: Prove* $P \to Q$ *and* $Q \to P$ *separately*

This makes total sense since by definition, $P \leftrightarrow Q$ means $P \to Q \wedge Q \to P$

Remark 31 ***To use a given of the form*** $P \leftrightarrow Q$ *: Treat* $P \to Q$ *and* $Q \to P$ *as two separate givens.*