

# Multivariate Functions

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## 1 Review

In economics, functions typically include several variables

$$U(x_1, x_2, \dots, x_n)$$
$$y = f(L, K)$$

For 3 variable case, can use *level curves* to illustrate functions.

**Example 1** *Draw an indifference curve*

**Example 2** *Draw an isoquant*

**Definition 3** *Partial derivatives* tell us how a function changes when one variable changes and all other variables are held constant.

Note the use of  $\partial$  rather than  $d$  in taking the derivative. The use of  $\partial$  tells us that we are taking a partial derivative.

**Definition 4** Consider  $U = U(x_1, x_2, \dots, x_n)$ . **Marginal utility:**  $MU_1 = U_1 = \frac{\partial U}{\partial x_1}$  is the partial derivative of the utility function with respect to  $x_1$ . It tells us how utility changes as only consumption of good 1 changes, consumption of all other goods held constant. So  $MU_1$  is the slope of a utility function where  $x_2$  has been held fixed at a certain value.

**Example 5** Draw  $U(x_1, x_2)$  with  $x_2$  held constant.

**Definition 6** Consider  $y = f(K, L)$   $MP_K = y_K = \frac{\partial y}{\partial K}$  is the partial derivative of the production function with respect to capital. It tells us how output changes when we change capital and hold labor fixed. So  $MP_K$  is the slope of the production function where labor has been held fixed at a certain value.

**Example 7** Draw  $y = f(K, L)$  with  $L$  held constant.

**Example 8** Calculate partial derivatives of  $f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$ .

### Example 9

$$\begin{aligned}Y &= C + I_0 + G_0 \\C &= \alpha + \beta(Y - T), (\alpha > 0, 0 < \beta < 1) \\T &= \gamma + \delta Y, (\gamma > 0, 0 < \delta < 1)\end{aligned}$$

*How does  $Y$  change when government spending changes?*

Second partials are analogous to second derivatives for functions of 1 variable.

- $\frac{\partial^2 U}{\partial x_1^2}$  tells us how fast utility changes when only  $x_1$  changes. It tells us how  $MU_1$  is affected by a change in  $x_1$
- $\frac{\partial^2 U}{\partial x_2^2}$  tells us how fast utility changes when only  $x_2$  changes. It tells us how  $MU_2$  is affected by a change in  $x_2$
- $\frac{\partial^2 U}{\partial x_1 \partial x_2}$  is a cross-partial. It tells us how  $MU_1$  changes when  $x_2$  changes. Note that  $\frac{\partial^2 U}{\partial x_1 \partial x_2} = \frac{\partial^2 U}{\partial x_2 \partial x_1}$

**Example 10**  $y = AK^{1-\alpha}L^\alpha$  where  $0 < \alpha < 1$  and  $K, L, A > 0$ . Calculate  $\frac{\partial y}{\partial K}$ ,  $\frac{\partial^2 y}{\partial K^2}$ ,  $\frac{\partial^2 y}{\partial L \partial K}$ . Sign and interpret.

Review chain rule.

## 2 Level curves

Let  $y = f(x_1, x_2)$ . What can we say about the slope and curvature of level curves?

### 2.1 Slope

If  $f_2 \neq 0$ , then can write  $x_2 = x_2(x_1)$ . Use chain rule and calculate  $\frac{\partial y}{\partial x_1}$ .

$$\begin{aligned}\frac{\partial y}{\partial x_1} &= \partial f x_1 + \frac{\partial f}{\partial x_2} \frac{dy}{dx_1} \\ 0 &= f_1 + f_2 \frac{dy}{dx_1} \\ \frac{dy}{dx_1} &= -\frac{f_1}{f_2}\end{aligned}$$

Sign this.

## 2.2 Curvature

We draw indifference curves and isoquants as convex functions. Lets see what is required for convexity. As before, assume  $x_2 = x_2(x_1)$ . Convexity requires  $\frac{d^2 x_2}{dx_1^2} > 0$ . From before, we know

$$\frac{dx_2}{dx_1} = -\frac{f_1(x_1, x_2(x_1))}{f_2(x_1, x_2(x_1))} \quad (1)$$

Calculate  $\frac{d^2 x_2}{dx_1^2}$ .

Thus, convexity depends on first, second, and cross partials .

## 2.3 Monotonic functions

**Definition 11** Consider  $V(x_1, x_2) = F(U) = F[U(x_1, x_2)]$ . If  $F'(U) > 0$  than  $V$  is a **monotonically increasing (monotonic) function of  $U$** .

**Example 12** *If we transform  $U = U(x_1, x_2)$  through a monotonic function  $V$ , what can you say about the slope of the new indifference curves? What conclusions can you draw about optimizing  $U$  versus  $V$ ?*

**Example 13** *Test this with the function  $U = x_1x_2$  and  $V = \ln x_1 + \ln x_2$ .*

We often exploit this idea to simplify problems.

**Example 14** Calculate the FOC for  $U = x_1^{\frac{1}{4}}x_2^{\frac{3}{4}}$  s.t.  $m = p_1x_1 + p_2x_2$ . Compare to the case of  $\ln U$ .

**Example 15** Lets examine the case of MU. Reconsider  $V = F[U(x_1x_2)]$  where  $F'(U) > 0$ . Suppose  $U$  exhibits diminishing MU. What conclusions can you draw about  $V$ ?

**Definition 16** With **ordinal** functions, ordering matters but actual index doesn't. With **cardinal** functions, the index number has meaning.

If a monotonic transformation preserves a property then that property is considered ordinal. Cardinal properties are not preserved by monotonic transformations. For this reason, economists prefer to deal with MRS rather than MU. MRS is an ordinal concept while MU is a cardinal concept. This explains what we observed above.

### 3 Two special types of functions

There are two special types of functions that arise a lot in economics: homogeneous and homothetic functions.

#### 3.1 Homogeneous functions

You see many types of functions in economics that are homogeneous, such as profit functions, cost functions, and demand functions.

**Definition 17** A function  $f(x_1, \dots, x_n)$  is **homogeneous of degree  $r$  (hodr)** iff  $f(tx_1, \dots, tx_n) \equiv t^r f(x_1, \dots, x_n)$ . In other words, increasing all variables by a constant amount  $t$  causes the value of the function to change by  $t^r$ .

**Example 18** What is the degree of homogeneity of  $f(x_1, x_2) = x_1x_2 + x_2^2$ ?

This is an example of an IRTS function ( $r > 1$ )

**Example 19** What is the degree of homogeneity of  $f(x_1, x_2) = \frac{x_1^2}{x_1x_2 - x_2^2}$ ?

**Example 20** Lets consider a more general version, which has the same property:

$$\text{Max}U(x_1, x_2) \text{ s.t. } p_1x_1 + p_2x_2 \tag{2}$$

**Example 21** Calculate the degree of homogeneity of the Cobb-Douglas production function,  $y = L^\alpha K^{1-\alpha}$ .

This is an example of CRTS. Lets consider this graphically.

*Calculate the degree of homogeneity of MPL. What general observation can you draw?*

We'll now show that this property holds for all functions.

### **3.1.1 Properties of homogeneous functions**

**Theorem 22** *If  $f(x_1, \dots, x_n)$  is  $h$ odr then the first partials  $f_1, \dots, f_n$  are  $h$ od( $r-1$ ).*

**Example 23** *Prove the above theorem*

**Theorem 24** *If  $f(x_1, \dots, x_n)$  is homr then the slopes of the level curves are unchanged along any ray from the origin.*

**Example 25** *Prove the above theorem*

This property has two important implications:

1. The income expansion path for a homogeneous utility function is a ray from the origin

2. Income elasticity of demand = 1

**Theorem 26** *Euler's Theorem: Suppose  $f(x_1, \dots, x_n)$  is hom. Then  $\frac{\partial f}{\partial x_1}x_1 + \dots + \frac{\partial f}{\partial x_n}x_n \equiv rf(x_1, \dots, x_n)$ .*

**Example 27** *Prove the above theorem*

**Example 28** *Let  $f(x_1, x_2) = x_1x_2 + x_2^2$ . Show that Euler's Theorem holds for this function.*

**Example 29** *Show that a CRTS firm earns zero profits.*

While these are helpful properties, homogeneity is a cardinal property. Thus, a monotonic transformation can make a function not homogeneous. But, some of the properties that we like about homogeneous functions (e.g., constant slope along a ray) intuitively seem like they should be cardinal; they should not depend on the value of a particular function. Does a more general class of functions have these same properties? Yes!

## 3.2 Homothetic functions

**Definition 30** *A homothetic function is a positive monotonic transformation of a homogeneous function.*

Homotheticity is an ordinal property because a monotonic transformation of a homothetic function is still homothetic. Thus, if  $y = f(x_1, \dots, x_n)$  is *hodr*, and  $z = F(y)$  where  $F'(y) > 0$ , then  $z$  is a homothetic function.

**Example 31** *Prove that a homothetic function has a constant slope along an expansion ray.*