Logic and Mathematical Notation*

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1 Overview

In this section, we will introduce some basic mathematical notation that will allow us to express arguments mathematically. We will begin translation and evaluation of arguments into truth tables. Having used truth tables to motivate an explanation of the criteria for a valid arguement, we will develop a method for doing proofs. We will use the equivalencies learned in the truth tables to do proofs.

2 Deductive reasoning and logical connectives

In this section, you will learn:

- How to identify premises and conclusions
- The criteria for determining whether a conclusion is valid
- Basic mathematical notation

Consider the following example.

Example 1 *It will either rain or snow tomorrow. It is too warm for snow. Therefore, it will rain.*

The first two sentences are *premises.* The last sentence is the *conclusion.*

Definition 2 *An arguement is valid if: when the premises are true the conclusion must also be true.*

Exercise 3 *Either Miss Scarlet is guilty or Mr. Green is guilty. Either Mr. Green is guilty or Mrs. Peacock is guilty. Therefore, either Miss Scarlet or Mrs. Peacock is guilty.*

1. Underline the premises.

*Excerpted liberally from Velleman [2006]

2. *Circle the conclusion.*

3. *Is the above a valid arguement? Why or why not?*

Result: an arguement is invalid if we can show that it is possible that when the premises are true, the conclusion can be false. This is a very important point and will guide our thinking on how to do proofs, so it is worth repeating:

- if when the premises are true, the conclusion is also true, an argument is valid
- if when the premises are true, the conclusion is false, an argument is invalid.
- The above example was therefore an invalid arguemnt.

So far, we've been writing everything out using words. Reducing complex statements to letters and symoblic notation, allows us to strip things down just to the arguement. Then we can judge a statement by its arguement without getting distracted by the phrasing.

Reconsider Example 1. Suppose we let P represent the possibility of rain and Q represent the possibility of snow. Then this argument has the form:

$$\begin{aligned} &P \text{ or } Q \\ &\text{not } Q \\ &\text{Therefore, } P. \end{aligned}$$

Just as we can use letters like P and Q to represent concepts, we can also introduce some other symbolic notation.

Table 1: Basic logic notation

Symbol	**Meaning**
$\vee$	or
$\wedge$	and
$\neg$	not

Therefore, $P \vee Q$ means "P or Q'' and the negation of P is $\neg P$.

Exercise 4 *Write the following in logical notation:*

1. *Joe is going to leave home and not come back.*

2. *Either Bill is at work and Jane isn't or Jane is at work and Bill isn't.*

3 Truth Tables

Recall that an arguement is valid if the premises cannot all be true without the conclusion also being true. Truth tables can be used to analyze complex arguements.

In this section, you will learn:

- How to construct truth tables
- How to use truth tables to evaluate arguements

Table 2: Truth table for $P \wedge Q$

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

Table 2 shows the truth table for $P \wedge Q$. It shows all the possible combinations of P and Q; each can either be true or false. We see that if P is false (ie, doesn't occur) and Q is false, then $P \wedge Q$ must also be false. Similarly, if just one of the elements is false, $P \wedge Q$ must be false. Only if both are true can $P \wedge Q$ be true.

Tables 3 and 4 shows the truth tables for $\neg P$and $P \vee Q$.

Table 3: Truth table for $\neg P$

P	$\neg P$
F	T
T	F

Table 4: Truth table for $P \vee Q$

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

Exercise 5 *Make the truth table for*

1. $\neg(P \vee \neg Q)$

2. $\neg(P \wedge \neg Q) \vee \neg R$

Think back to Example 1 from earlier. Make a truth table and use it to analyze whether the arguement is valid. Recall that we decided that the form of this arguement was:

$$\begin{array}{rl} & P \vee Q \\ & \neg Q \\ \therefore & P \end{array}$$

We said that the requirement for an argument to be valid is that if the premises are all true, the conclusion must also be true. The premises here are $(P \vee Q)$ and $\neg Q$. The conclusion is P.

P	Q	$P \vee Q$	$\neg Q$
F	F	F	T
F	T	T	F
T	F	T	T
T	T	T	F

Observe that when the premises are both true, the conclusion is also true. Therefore, we have shown symbolically that this arguement is valid.

Exercise 6 *Use a truth table to determine whether the following arguement is valid:*

1. *Either John isn't stupid and he is lazy, or he's stupid. John is stupid. Therefore, John isn't lazy.*

2. *The butler and the cook are not both innocent. Either the butler is lying or the cook is innocent. Therefore, the butler is either lying or guilty.*

3.1 Equivalencies

Many times, logical statements can be simplified, making the truth tables easier and faster to calculate. This is because symbolic logic follows many of the same properties as typical math.

- DeMorgan's Law: $\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$
- Commutative laws: $P \wedge Q$ is equivalent to $Q \wedge P$
- Associative laws: $P \wedge (Q \wedge R)$ is equivalent to $(P \wedge Q) \wedge R$
- Idempotent laws: $P \wedge P$ is equivalent to P
- Distributive laws: $P \wedge (Q \vee R)$ is equivalent to $(P \wedge Q) \vee (P \wedge R)$
- Double negation law: $\neg\neg P$ is equivalent to P

You can (and should!) show that the above are equivalent.

Exercise 7 *1. Show DeMorgan's Law*

2. Find a simpler equivalent formula: $\neg(P \vee \neg Q)$

3. Find a simpler equivalent formula: $\neg(Q \wedge \neg P) \vee P$

Formulas that are always true, such as $P \vee P$, are called *tautologies*. Formulas that are always false are called *contradictions*.

4 Variables and Sets

In this section, you will learn:

- How to incorporate a variable into a statement
- Some basics about sets

So far, we've symbolized statements using simple letters. We have not accounted for the fact that a statement might include variables. Consider the statement, "x is a prime number". Let x be a variable. We can write this statement as $P(x)$. By writing this in this way, rather than just P, we are accounting for the fact that x can take on different values. If a statement includes more than one variable, all variables should be shown: eg, $D(x, y)$.

Exercise 8 *Write out the following in symbolic notation*

1. *x is a man and y is a woman and x likes y but y does not like x.*

2. *x is a prime number, and either y or z is divisible by x.*

With dealing with statements without variables, a statement is always either true or false. This is not the case when a statement includes variables; the truth of a statement depends on the value that the variable takes. From our example before, $P(7)$ is true while $P(9)$ is false. We deal with this through the introduction of *truth sets.*

Definition 9 *A truth set is the set of values of x for which $P(x)$ is true.*

Definition 10 *A set is a collection of items.*

Definition 11 *The objects in the set are called elements of the set.*

Example 12 $A = \{3, 7, 14\}$

$\in$ means "is an element of". Therefore, in the above example, $7 \in A$ but $11 \notin A$. The order of elements in a set doesn't matter.

Example 13 *If* $B = \{14, 7, 3\}$ *then* $A = B$.

It can be tiresome to write out all the numbers in a set. We have notation that simplifies this. $C = \{x | \ x \text{ is a prime number}\}$ is read as "C=the set of all x such that x is a prime number". So, C are the values of x that make the statement "x is a prime number" true. So the statement "x is a prime number" is an *elementhood test* Any value of x that make the statement come out true passes the test and is an element of the set. Therefore, the truth set is the set that passes the elementhood test.

Example 14 *n is an even prime number. The truth set is 2.*

There are certain sets that are frequently referred to in mathematics.

Definition 15 $R = \{x | \ x \textit{ is a real number}\}$. *A real number is any number on the number line. So R consists of integers, the rational numbers, and all of the other numbers on the number line that don't meet the above defintions, such as* $\sqrt{2}$.

Definition 16 $Q = \{x | \ x \textit{ is a rational number}\}$. *A rational number can be written as a fraction,* $\frac{p}{q}$, *where p and q are integers.*

Definition 17 $Z = \{x | \ x \textit{ is an integer}\}$. *Integers are the natural numbers, their negatives, and 0. ie,.* $Z = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$

Definition 18 $N = \{x | \ x \textit{ is a natural number}\}$. *Natural numbers the numbers that we use for counting: ie.,* $N = \{0, 1, 2, 3, ...\}$

You can use subscript $+$ or $-$ to indicate positive or negative numbers (eg.,

$$R^+ = \{x | \ x \text{ is a positive real number}\}.$$

Go back to our example with C. "x is a prime number" is an *elementhood test* for the set. Any value of x that makes the statement true passes the test and thus is an element of the set. If it fails the test, it isn't an element of the set.

Example 19 $\{x | \ x^2 < 9\}$. *Since* $5^2 \not< 9$ *then* $5 \notin \{x | \ x^2 < 9\}$.

Exercise 20 *Translate*

$$a + b \notin \{x | \ x \textit{ is an even number}\}$$

Here is some more useful set notation.

Definition 21 *U (universe) is the set of all possible values that variables can take in a particular problem.*

Definition 22 *$\emptyset$ is the empty set or null set, meaning a set of no elements.* $\emptyset = \{\}$

Example 23 $\{x \in Z|\ x \neq x\} = \emptyset$

Just as with symbolic logic, we have operations on sets.

Definition 24 *Intersection:* $A \cap B = \{x|x \in A \text{ and } x \in B\}$

Definition 25 *Union:* $A \cup B = \{x|x \in A \text{ or } x \in B\}$

Definition 26 *Difference:* $A \backslash B = \{x|x \in A \text{ and } x \notin B\}$

Exercise 27 $A = \{1, 2, 3, 4, 5\}$ *and* $B = \{2, 4, 6, 8, 10\}$.

$$\begin{aligned} A \cap B &= \\ A \cup B &= \\ A \backslash B &= \\ (A \cup B) \setminus (A \cap B) &= \\ (A \backslash B) \cup (B \backslash A) &= \\ B \backslash A &= \end{aligned}$$

Exercise 28 $A = \{x | \ x \textit{ is a man}\}$. $B = \{x | \ x \textit{ has brown hair}\}$.

$$\begin{aligned} A \cap B &= \\ A \cup B &= \\ A \backslash B &= \end{aligned}$$

5 Conditional and Biconditional Connectives

A very important set of words in arguements are of the form "if...then". Example **??** shows this type of arguement. $P \to Q$ is read as "if P then Q". This is called a *conditional statement*. P is the *antecedent* and Q is the *consequent*. Consider the following example:

Example 29 ?? *If today is Sunday, then I don't have to go to work. Today is Sunday. Therefore, I don't have to work today.*

Putting these statements into logical form, we get:

$$\begin{aligned} P \quad \to & \quad Q \\ & \quad P \\ \therefore & \quad Q \end{aligned}$$

Exercise 30 *Put in logical form.*

1. *If it's raining and I don't have my umbrella, then I'll get wet.*

2. *If Mary did her homework, then the teacher won't collect it, and if she didn't then he'll ask here to do it on the board.*

You won't be surprised to find that we can also express conditional statements through truth tables. What we see in Table 5 is that a statement is false if the antecedent is true and the consequent is false; otherwise the statement is true. We'll write this out and the go through Example **??** to see why this is the case.

Table 5: Truth table for $P \to Q$

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

If you get 90% on the final, you'll pass.

- Student gets 90% on the final (P is true) and he passes the class (Q is true). $P \to Q$ is true.
- Student gets 90% on the final (P is true) and he fails the class (Q is false.). $P \to Q$ is false.
- Student gets 70% on the final (P is false) and he fails the class (Q is true). $P \to Q$ still holds. The assumptions were not met.
- Student gets 70% on the final (P is false) and he passes the class (Q is true). $P \to Q$ is still true as it has not been dis-proved.. My passing him was not a lie as I didn't say 'only if'.

Exercise 31 *Now do the truth table for $\neg P \vee Q$ and see what you notice.*

Hopefully you see that the column under $\neg P \vee Q$ and $P \rightarrow Q$ are the same! This tells us that the two statements are equivalent since they have the same truth values. (From before, we know that one can also rewrite $\neg P \vee Q$ to find other equivalencies.)

An intutive example of why the two are equivalent can be seen from the statement: "You won't not clean your room again or you'll be grounded".

Exercise 32 *Use a truth table to analyze the validity of the following arguement. "If Jones was convicted of murdering Smith, then he will go to jail. Jones will go to jail. Therefore, Jones was convicted of murdering Smith."*

5.1 Contrapositive versus converse

Contrapositive and converse are not the same thing. The *converse* of the statement $P \rightarrow Q$ is $Q \rightarrow P$. They are not equivalent. (You can confirm this by checking their truth tables.) You can also see this intuitively. The statement "If you have a PhD then you are smart" is different from "If you are smart then you have a PhD".

The *contapositive* of $P \rightarrow Q$ is $\neg Q \rightarrow \neg P$. The two statements are equivalent. You can verify this with a truth table. Again, we can see this intuitively. The statement "If you live in Albuqerque, then you live in New Mexico" is the same as "If you don't live in New Mexico then you don't live in Albuquerque".

Exercise 33 *Show that the contrapositve and converse have different truth tables.*

Exercise 34 *Which of the following statements are equivalent?*

1. *If it's either raining or snowing, then the game has been canceled.*

2. *If the game hasn't been canceled, then it's not raining and it's not snowing.*

3. *If the game has been canceled then it's either raining or snowing.*

It turns out that there are a number of different ways of saying $P \to Q$.

- If P then Q
- P implies Q
- P only if Q
  - Consider the sentence "You can vote in US national elections only if you are a US citizen". In other words, if you aren't a US citizen, you can't vote in US national elections. This is $\neg Q \to \neg P$ which we know is equivalent to $P \to Q$.
- P is a sufficient condition for Q
- Q is a necessary condition for P
  - This means that if Q doesn't occur than P can't occur. In other words, $\neg Q \to \neg P$ or $P \to Q$.

Exercise 35

Analyze the logical forms of the following sentences

1. *If at least 10 people are there, then the lecture will be given*

2. *The lecture will be given only if at least 10 people are there*

3. *The lecture wil be given if at least 10 people are there*

4. *Having at least 10 people there is a sufficient condition for the lecture being given.*

5. *Having at least 10 people there is a necessary condition for the lecture being given.*

5.2 A last bit of notation

$P \longleftrightarrow Q$ is a *biconditional statement* and it is shorthand for $(P \to Q) \wedge (Q \to P)$. This is translated as "P if Q and P only if Q". More often you'll see this expressed as "if and only if" or *iff.* Later on, we'll often see this expressed as "P is a necessary and sufficient condition for Q".

Exercise 36 *Analyze the logical forms of the following sentences.*

1. *The game will be canceled iff it's either raining or snowing*

2. *Having at least 10 people there is a necessary and sufficient condition for the lecture being given.*

3. *If John went to the store then we have some eggs, and if he didn't then we don't.*

6 Quantificational Logic

6.1 Quantifiers

A little more notation:

- $\forall$ means "for all"
  - ex) $\forall x P(x)$ means that "for all values of x, $P(x)$ is true"
- $\exists$ means "there exists"
  - ex) $\exists x P(x)$ means that "there exists a value of x, such that $P(x)$ is true"

Words like everyone, someone, everything, or something are signals that you will need a quantifier.

Exercise 37 *What do the following mean? Are they true or false?*

1. $\forall x\left(x^2 \geq 0\right)$, *where the universe is* R

2. $\exists x\left(x^2 - 2x + 3 = 0\right)$, *where the universe is* R

3. $\exists x\left(M(x) \wedge B(x)\right)$, *where the universe is the set of all people,* $M(x)$ *stands for the statement "*x *is a man" and* $B(x)$ *means "*x *has brown hair".*

4. $\forall x\left(M(x) \rightarrow B(x)\right)$ *with the same universe and meanings as above*

Exercise 38 *What do the following mean? Are they true or false? The universe is* N*,natural numbers.*

1. $\forall x \exists y\,(x < y)$

2. $\exists y \forall x\,(x < y)$

3. $\exists x \forall y\,(x < y)$

Exercise 39 *Analyze the logical form of the following statements*

1. *Everybody in the dorm has a roomate he doesn't like.*

2. *If anyone in the dorm has a friend who has the measles, then everyone in the dorm will have to be quarantined.*

Note: all mathematical statements can be understood using the 7 logical symbols that we have so far introduced: $\vee, \wedge, \neg, \rightarrow, \leftrightarrow, \exists, \forall$.

References

Daniel J. Velleman. *How to Prove It: A Structured Approach.* Cambridge University Press, Cambridge, second edition, 2006.