Production, Costs, and Supply

March 8, 2004

Having studied the underlying model of consumer demand, we now move to studying how firms make decisions.

1 Production Functions

Definition 1 *A production function is a mathematical relationship between inputs and outputs.*

$$q = f(K, L, M, ...)$$

where

$$\begin{aligned} q &= \textit{output} \\ K &= \textit{capital} \\ L &= \textit{labor} \\ M &= \textit{raw materials.} \end{aligned}$$

A production function tells us how inputs are converted into output. Just as in the case of a utility function where we simplified things by assuming just two goods, x_1 and x_2, we tyically simplify a production function by assuming just two inputs, K and L.

$$q = f(K, L)$$

There are lots of everyday examples of production functions:

- Home production (fixing a leak in the roof, child care, cooking, etc).
- Health: People combine purchased inputs such as medicine or medical services with their own time to produce health.

1.1 Marginal Product

Recall that marginal means small change. Therefore, marginal product is the change in output when an input is changed. We have two types of marginal products: marginal product of labor and marginal product of capital. Not surprisingly, each can be represented through a mathematical calculation involving the derivative. Because our production function is a function of two variables, K and L, we'll be dealing with partial derivatives.

1.1.1 Marginal Product of Labor

Definition 2 *Marginal Product of Labor (MP_L): tells us how output changes when the amount of labor is changed.*

$$MP_L = \frac{\partial q}{\partial L}$$

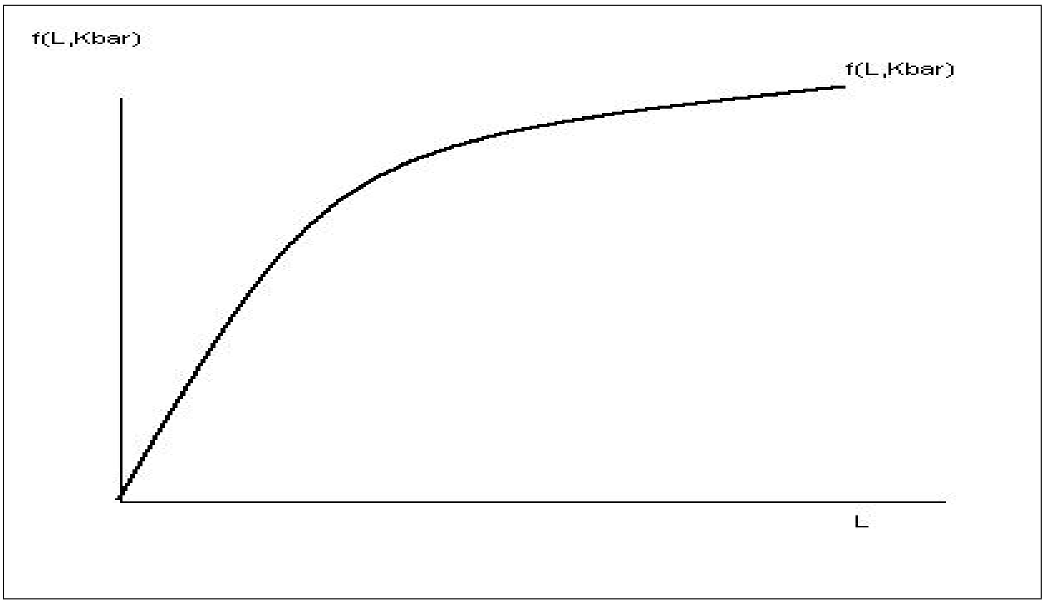
We would anticipate that $MP_L > 0$; as the number of workers increase, output should increase. However, it seems reasonable that output might increase at a decreasing rate. In other words, while each additional worker increases the amount of output, he might add less output than the previous worker. We call this dimininishing MP_L. Because it tells us how the change in output changes when the number of workers change, diminishing MP_L is represented by a derivative. It is the derivative of the MP_L function meaning it is the second derivative of the production function.

Definition 3 *Diminishing Marginal Product of Labor: Each additional worker contributes a smaller increase in output.*

$$\frac{\partial MP_L}{\partial L} = \frac{\partial^2 q}{\partial L^2} < 0$$

Intuitively, diminishing MP_L occurs because when you add additional workers but keep capital fixed, the worker can not be as productive as the previous worker because he has to wait around to get a chance to use the machine.

Figure 1: Production function where Capital is held fixed



We can see these concepts by looking at a graph. Consider a production function where we hold the amount of capital fixed. We want to know how much output will be produced for different amounts of labor, given a fixed amount of capital. Mathematically, we symbolize this as

$$q = f\left(\overline{K}, L\right).$$

The graph below illustrates this production function. Note that the slope of the production function is positive but gets smaller as we increase the amount of labor. This says what we said above mathematically. MP_L is positive and diminishing.

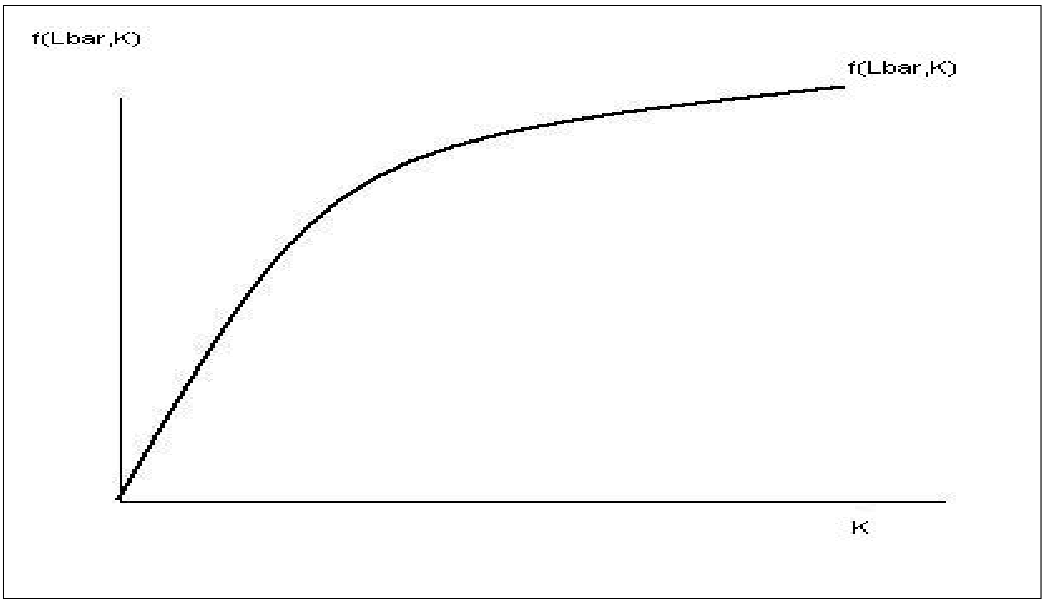
1.1.2 Marginal Product of Capital

Definition 4 *Marginal Product of Capital (MP_K): tells us how output changes when the amount of capital is changed.*

$$MP_K = \frac{\partial q}{\partial K}$$

We would anticipate that $MP_K > 0$; as the number of workers increase, output should increase. However, it seems reasonable that output might increase at a decreasing rate. In other words, while each additional machine increases the amount of output, he might add less output than the previous machine. We call this dimininishing MP_K. Because it tells us how the change in output changes when the number of machines change, diminishing MP_K is represented by a derivative. It is the derivative of the MP_K function meaning it is the second derivative of the production function.

Figure 2: Production function where Labor is held fixed



Definition 5 *Diminishing Marginal Product of Capital: Each additional machine contributes a smaller increase in output.*

$$\frac{\partial MP_K}{\partial K} = \frac{\partial^2 q}{\partial K^2} < 0$$

Intuitively, diminishing MP_K occurs because when you add additional machines but keep the number of workers fixed, the machine can not be as productive as the previous machine because the machine is being used less frequently.

We can see these concepts by looking at a graph. Consider a production function where we hold the amount of labor fixed. We want to know how much output will be produced for different amounts of capital, given a fixed amount of labor. Mathematically, we symbolize this as

$$q = f\left(K, \overline{L}\right).$$

The graph below illustrates this production function. Note that the slope of the production function is positive but gets smaller as we increase the amount of capital. This says what we said above mathematically. MP_K is positive and diminishing.

1.2 Three dimensional depiction of a production function

The above discussion tells us something about the shape of the production function when one variable is held fixed. But in reality, we know that typically when you change one variable, you'll also change the other. In other words, if you buy more machines, it is likely that you'll hire more workers to use them at the same time and vice versa. We want to think about how to depict the production function when both variables are allowed to change.

The graph above shows this idea. It is the exact same idea as our three-dimensional utility function. On the x axis we have labor, on the y axis we have capital, and on the z axis we have the amount of output. The graph shows us for any combination of capital and labor, how much output we will produce.

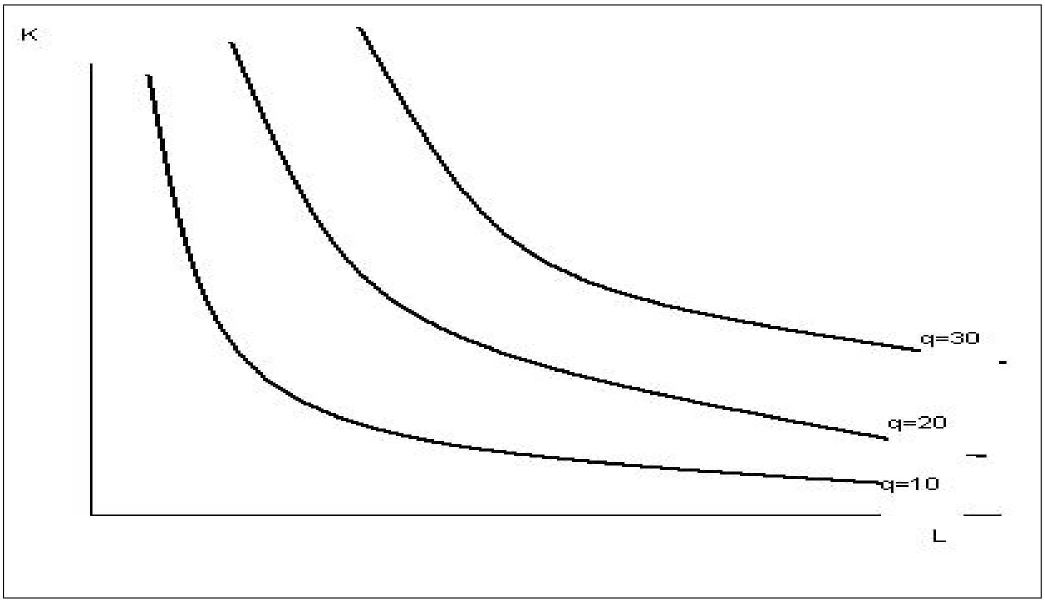
2 Isoquants

The 3 dimensional graph is a true depiction of a production function but what a pain to draw! Just as with our utility functions, we would like to depict the production function using a two dimensional graph. Analgous to indifference curves, we have isoquants.

Definition 6 *An isoquant shows all the difference combinations of labor and capital that can be used to produce a certain level of output.*

The graph above shows some example isoquants. Just as with indifference curves, these isoquants are levels curves and come from the production fucntion. In other words, we take a 3 dimensional production function and slice it at different production levels and trace these slices onto the capital/labor plane. The isoquants are labeled with their production level. Note the shape of the isoquants. Note that increasing the amount of both labor and capital results in a higher production level; thus, as you move in a northeasterly direction, isoquants are labeled with higher production levels. In contrast to indifference curves, the number that labels an isoquant has a meaning. It has a production function. Remember that with indifference curves the actual labeling didn't matter; just the order of labels mattered. In other words, when you moved from one indifference curve labeled

Figure 3: Example Isoquants



$U = 10$ to another indifference curve labeled $U = 20$, the fact that one indifference curve was 10 utils higher than the other didn't mean anyhing. In contrast, if we're talking about isoquants and one is labeled $q = 10$ and the other is labeled $q = 20$, the fact that one isoquant is 10 units higher than the other has meaning. It tells us that more 10 more units are being produced.

2.1 Marginal rate of technical substitution (MRTS)

The MRTS tells you that rate at which you can you trade-off between two inputs and keep output constant.

Definition 7 *MRTS: The amount by which one input can be reduced when another input is added while holding output constant.*

You guessed it - this description tells us what happens as we move along an isoquant. Therefore, the MRTS is the slope of the isoquant. Mathematically,

$$MRTS = -\frac{MP_L}{MP_K}$$

Essentially, this is derived in the following way:

$$
\begin{aligned}
MP_L * \Delta L + MP_K * \Delta K &= 0 \\
\frac{\Delta K}{\Delta L} &= -\frac{MP_L}{MP_K} \\
\frac{dk}{dL} &= -\frac{MP_L}{MP_K} \\
MRTS &= -\frac{MP_L}{MP_K}.
\end{aligned}
$$

Is an isoquant positively or negatively sloped? Note that

$$
\begin{aligned}
MP_L &> 0 \\
MP_K &> 0.
\end{aligned}
$$

Therefore, $MRTS < 0$. The formula for $MRTS$ also helps us understand the shape of the isoquant. If you are at point A, using a lot of K and a little L, one can give up a lot of K in exchange for a small increase in L, and still produce $q = 10$. If you are at point B, using a lot of L and a little K, one must give up a little K in exchange for a large increase in L, and still produce $q = 10$. Therefore, we say that these isoquants exhibit diminishing $MRTS$.

2.2 Input Substitution

How easily can you substitute one input for another? This is analgous to the question of how willing you are to substitute consumption of one good for another. So this question can be answered by looking at the shape of the isoquants.

2.2.1 Fixed Proportions

Consider the case where no substitution is possible. Output can only be produced by combining a certain amount of labor and capital. One example would be mowing a lawn. Every lawnmower needs a person; having more of one without the other would not increase production. We already know what these isoquants should look like.

2.2.2 Perfect Substitutes

The opposite extreme is where two inputs are perfect substitutes for each other. Capital and labor are equally good at producing a certain output. No surprises here as to what the isoquants look like.

2.2.3 The middle ground in between

Most goods neither require a fixed proportion of capital to labor nor are capital and labor perfect substitutes. The typical isoquant looks like the ones above.

Figure 4: Fixed Proportions Isoquants

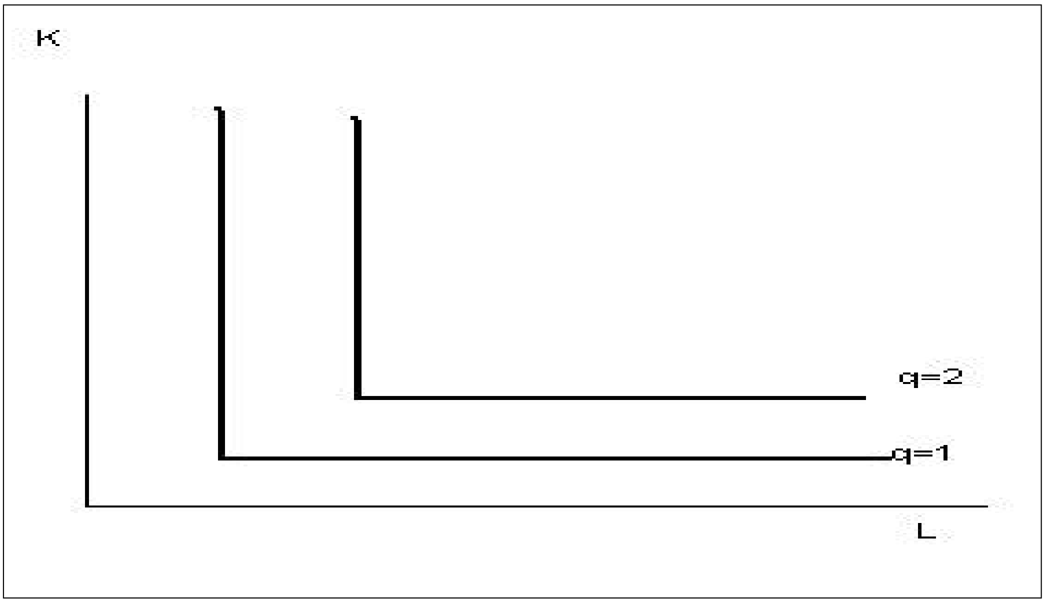
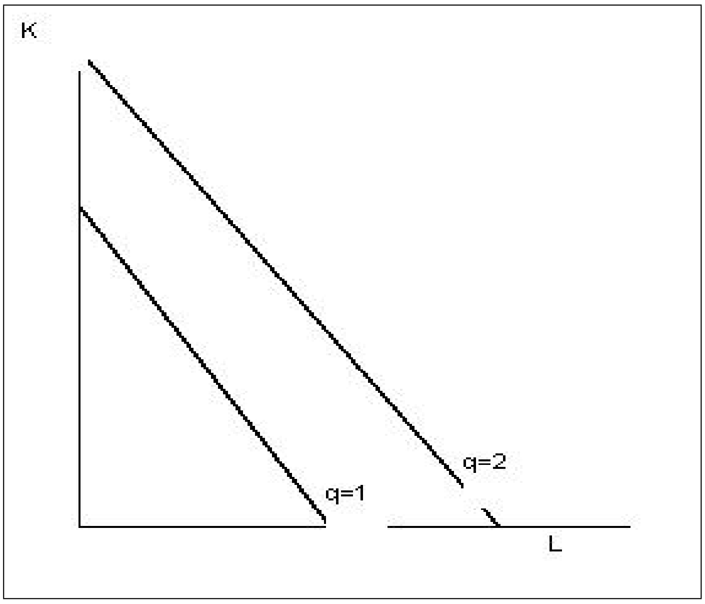


Figure 5: Perfect Substitutes Isoquants



2.2.4 Example: Speed Limits and Gas Consumption

Suppose person drives 6000 miles/year between home and work. Speed affects amount of time spent commuting and amount of gas used. Gas and time are inputs into transportation. There is a trade-off between the two. A faster speed implies less time but more gas. A slower speed implies more time but less gas.

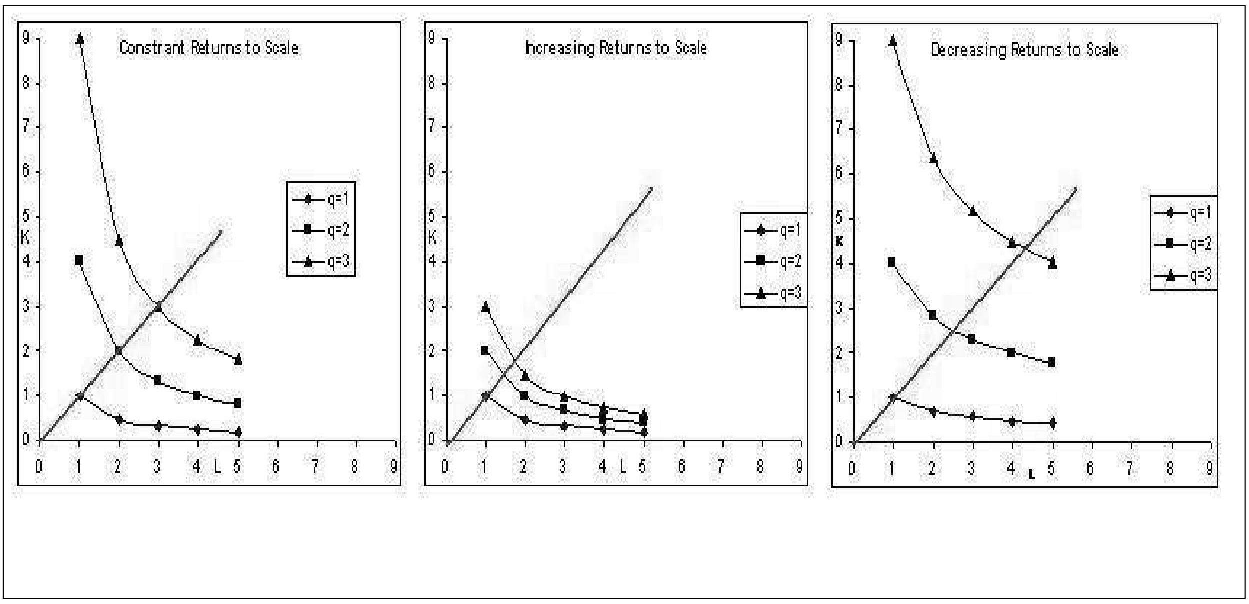
Both A and B are technologically efficient. They are different ways of obtaint the same output of commuting 6000 miles.Which point is better? Knowing this requires knowing more information about relative costs of inputs. Like with the conusmer problem, knowing this will require something like a budget constraint.

2.3 Returns to Scale

Definition 8 *Returns to scale: The rate at which output increases in response to proportional increases in all inputs.*

If you double all inputs, does output double, more than double, or less than double? This is the question that returns to scale attempts to answer. Lets first look at this issue graphically and then mathematically.

Figure 6: Returns to Scale



Definition 9 *Constant Returns to Scale (CRTS): Doubling the inputs causes output to double.*

The first graph shows CRTS. When you use 1 unit of K and one unit of L, $q = 1$. Doubling the inputs to $K = 2$ and $L = 2$ results in $q = 2$.

Definition 10 *Increasing Returns to Scale (IRTS): Doubling the inputs causes output to more than double.*

The second graph shows IRTS. When you use 1 unit of K and one unit of L, $q = 1$. Doubling inputs to $K = 2$ and $L = 2$ would give you $q > 2$.To get $q = 2$, you can less than double K and L.

Definition 11 *Decreasing Returns to Scale (DRTS): Doubling the inputs causes output to less than double.*

The second graph shows DRTS. When you use 1 unit of K and one unit of L, $q = 1$. Doubling inputs to $K = 2$ and $L = 2$ would give you $q < 2$.To get $q = 2$, you need to more than double K and L.

We can show this same concept mathematically.

Example 12 $q(K, L) = KL$. *If you double K and L,*

$$\begin{aligned} q(2K, 2L) &= (2K)(2L) \\ &= 4KL \\ &= 4q \end{aligned}$$

In this example, doubling inputs caused output to more than double. So this is IRTS.

We can generalize this idea by talking about increasing all inputs by some proportional factor t. For example, if $t = 2$ then we're doubling inputs. If $t = 3$ then we're tripling inputs.

Example 13 $q(K, L) = KL$

$$\begin{aligned} q(tK, tL) &= (tK)(tL) \\ &= t^2 KL \\ &= t^2 q \end{aligned}$$

Since output has increased by more than the factor t (it has increased by a factor of t^2), this is IRTS.

Definition 14 *Returns to Scale: Consider increasing all inputs by a factor of t :*

$$f(tK, tL) = t^r f(K, L)$$

If $r > 1$ then we have IRTS. If $r < 1$ then we have DRTS. If $r = 1$ then we have CRTS.

Example 15 $q = K^{\frac{1}{2}} L^{\frac{1}{2}}$

Example 16 $q = K^2L^2$

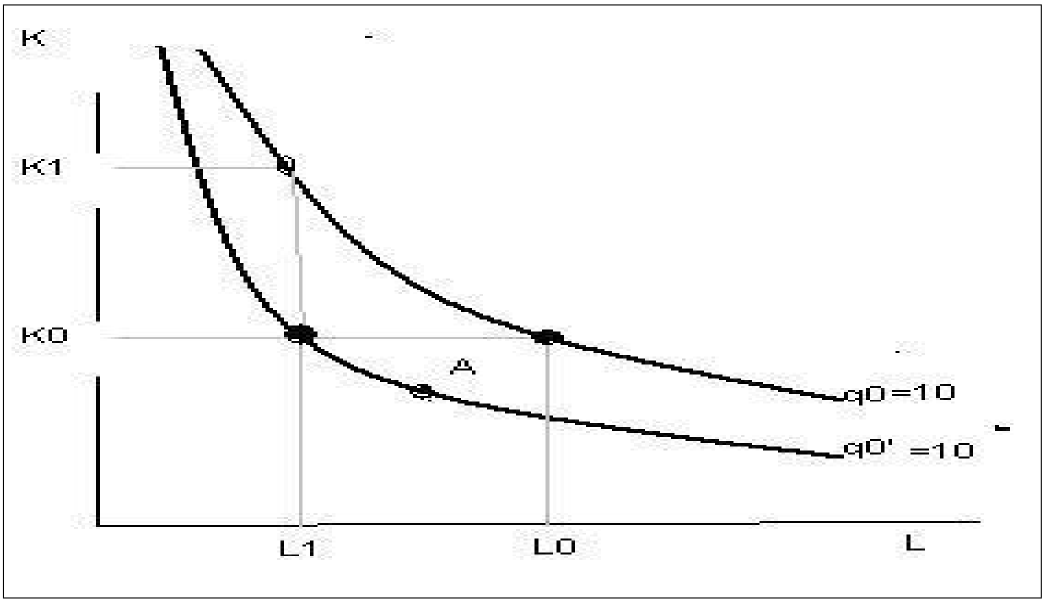
Example 17 $q = K^{\frac{1}{3}}L^{\frac{1}{3}}$

3 Changes in Technology

A production function reflects firms' technical knowledge about how to use input to produce outputs. As the productive technology improves, this means that can produce the same output with fewer inputs. Graphically this looks like a leftward shift of the isoquants. With the original technology, q_o could be produced using inputs L_0 and K_0. With the new technology, the same level of ouput (designated $q_0^{'}$) can be made using the same level of capital K_0 but a lower level of labor, L_1. For example, better trained workers could use the same resources more efficiently. Fewer would be required to create the same level of output. A represents an example where the same level of output can be made using both less capital and labor.

Note the differences between this and input substitutability. One could have achieved output q_0 by using L_1 rather than L_o. But in the case of no technological change, this would have required an increase in the amount of capital to K_1. Note that if one just looked at output per labor, the change from input substitution and the change from a change in technology would have looked the same. Before, output per labor was $\frac{q_0}{L_0}$ and after, output per labor was $\frac{q_0}{L_1}$. One also needs to look at output per capital. In the first case, output per capital is $\frac{q_0}{K_1}$ and afterwards it is $\frac{q_0}{K_0}$.

Figure 7: Technological change



4 A numerical example

$$q = K^{\frac{1}{2}} L^{\frac{1}{2}}$$

Suppose $q = 10$.

Graph the isoquant.

What is MP_L? What is MP_K? Do they satisfy our typical assumptions? What is AP_L?

What type of returns to scale are there?

Suppose there is a technological change, such as to

$$q = 20K^{\frac{1}{2}}L^{\frac{1}{2}}$$

Graph the new isoquant.