1 Play it again Sam....This time with the math

In fact, many of the ideas and concepts that we have presented so far in the class, can be easily expressed through calculus. So lets go through the same material, this time using calculus.

1.1 The Derivative and Partial Derivative

Many times so far, we have used the word "marginal". "Marginal" is synonymous with the idea of a derivative.

Marginal refers to a small change; a derivative measures the change in one variable when another variable changes.

1.1.1 The Derivative

Lets first consider the case of a univariate (one variable) function. For example, suppose we were just interested in your utility from consuming one good only. A univariate function would be:

$$U = U(x).$$

We might be interested in how your utility changes as your consumption of that good changes. This is the idea of a derivative. The notation for this is:

$$U'(x) \text{ or } U' \text{ of } \frac{dU}{dx}$$

This tells us the marginal effect on your utility of slightly changing your consumption of the good x.

Definition 1 *The basic rule for calculating a derivative is as follows. Suppose* $U(x) = kx^{\alpha}$ *where* k *is a constant,* x *is the variable, and* α *is a constant. Then* $U'(x) = \alpha k x^{\alpha-1}$

Bring the exponent down and subtract one from the exponent.

Lets look at some examples.

Example 2 $U(x) = x^2$

Example 3 $U(x) = 2x^3$

Example 4 $U(x) = 2x^{\frac{1}{2}}$

In the context of utility, x must always be positive. What does this imply for the sign of $U'(x)$? What assumption of preferences does this satisfy?

What is the derivative of a constant? Suppose $U = 2$? How does U change as x changes? U doesn't change because there is no x to change.

Definition 5 *If* $U = k$ *then* $U' = 0$.

Definition 6 *If* $U(x) = f(x) + g(x)$ *then* $U'(x) = f'(x) + g'(x)$

The derivative of a sum of function is the sum of the derivatives of the functions.

Example 7 $U(x) = 3x + x^2$

1.1.2 The Partial Derivative

Consider a multivariate (multiple variable) function, specifically a two-variable function. This is what we have been considering so far:

$$U = U(x_1, x_2)$$

This says that your utility comes from consumption of two goods, x_1 and x_2. A partial derivative is conceptually very similar to a derivative; it tells us how your utility changes as **consumption of one good changes and your consumption of the other good stays the same**. Note the difference. A derivative deals with a univariate function. It says how your utility changes as that variable changes. A partial derivative deals with a multivariate functions. It tells us how your utility changes when just one variable changes and no other variables change (thus the modifier, partial).

The notation for a partial derivative is

$$\frac{\partial U}{\partial x_1} \text{ or } U_1.$$

This tells us how our utility changes if we change our consumption of x_1 but don't change our consumption of x_2. Notice in comparision with the regular derivative that the "d" is drawn as squiggly. Therefore, to represent how our utility changes when we change our consumption of x_2 but don't change our consumption of x_1, we would write

$$\frac{\partial U}{\partial x_2} \text{ or } U_2.$$

These are the marginal utilities that we introduced earlier. (marginal tells us derivative)

Definition 8 *If $U = kx_1^{\alpha}x_2^{\beta}$ where $k, \alpha,$ and β are constants, then*

$$\begin{aligned} U_1 &= \alpha k x_1^{\alpha-1} x_2^{\beta} \\ U_2 &= \beta k x_1^{\alpha} x_2^{\beta-1} \end{aligned}$$

We calculated U_1 by treating x_2 as if was a constant. It is as if $x_2^{\beta} = 2$. Nothing about this term changes when we take the partial derivative with respect to x_1. We take the partial with respect to x_1 in the same manner as we did the basic derivative.

Lets look at some examples.

Example 9 $U = x_1 x_2$

Example 10 $U = x_1^2 x_2^2$

Example 11 $U = x_1 + x_2$

1.2 Marginal Rate of Substitution

An important result from calculus which I will just state is the Implicit Function Theorem. The Implicit Function Theorem allows us to calculate how one variable changes when another variable changes.

Definition 12

$$\frac{dx_2}{dx_1} = -\frac{U_1}{U_2}$$

Recall that the MRS (the rate at which you are willing to trade off between x_1 and x_2) is expressed through the slope of the indifference curve. The slope of the indifference curve is $\frac{dx_2}{dx_1}$. From the implicit function theorem, we know that

$$\frac{dx_2}{dx_1} = -\frac{U_1}{U_2}$$

This is the result we obtained earlier: the slope of the indifference curve is the ratio of the marginal utilities.

1.2.1 Monotonic Transformations

Example 13 $U = 2x_1^2x_2$. *Calculate* U_1 *and* U_2 *and MRS.*

Example 14 $U = 4x_1^2x_2$. *Calculate* U_1 *and* U_2 *and MRS.*

Note that the second function is just a monotonic transformation of the first function (we have multiplied by 2). The marginal utility for each function is different; however, the MRS is the same. It tells us that even if we do a monotonic transformation, the rate at which we are willing to trade off between two goods is the same. This is the result we would expect since the two functions represent the exact same underlying preferences.

1.3 Applications

1.4 The Budget Constraint

Our budget constraint is

$$p_1x_1 + p_2x_2 = m$$

where p_1, p_2, m are constants and x_1 and x_2 are variables. We can solve this explictly for x_2 as a function of x_1 :

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1.$$

So here, we have a univariate equation.

In thinking about the slope of this function, we are interested in calculating $\frac{dx_2}{dx_1}$.

$$\frac{dx_2}{dx_1} = -\frac{p_1}{p_2}$$

Since prices are restricted to being positive, we must have

$$\frac{dx_2}{dx_1} < 0$$

Therefore, the budget line has a negative slope, just as we have been drawing it. More intuitively, this calculation tells us that if we increase our consumption of x_1 we must decrease our consumption of x_2. This makes sense as we have a limited amount of money to spend.

Furthermore, this equation tells us that when we increase x_1 by one unit, then x_2 must fall by the amount $\frac{p_1}{p_2}$. Everytime we get one more unit of x_1, we need to give up $\frac{p_1}{p_2}$ units of x_2. Using the derivative interpration gives us a nice interpretation for the slope of a budget line. Slope measures rate at which x_2 can be traded for x_1 in the market. Thus, slope of budget line represents opportunity cost. In order to consume more of x_1 have to give up some of x_2.

1.5 The Utility Maximization Problem

We now know the intution and graphical approach to how to solve the basic economic problem of choosing the best bundle that you can afford. While this approach is very useful conceptually it is easy to see how difficult it would be to solve for the optimal bundles quickly. To do that, we need to utilize calculus. One type of calculus problem is that of optimization; finding the set of variables that give the highest value subject to a set of constraints. This is in fact the the idea of our Utility Maximization problem: find the combination of goods that maximize our utility (function) subject to an income constraint.

1.5.1 Univariate function

To better understand the optimization approach, let's go back to considering the simplest utility function $U(x)$ and for the moment, ignore the budget constraint. So we would like to know how much x we should choose so as to maximize our utility. (For this example, assume that you can be satiated - ie, suppose that there is a limit beyond which more is not better.)

Lets first approach this problem graphically.

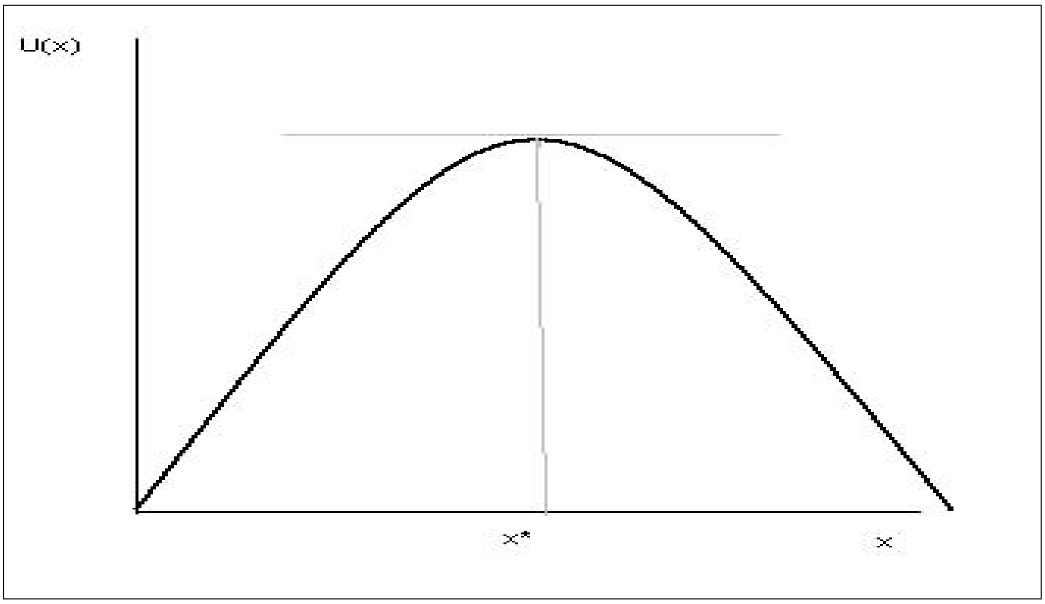
The graph above shows a univariate function $U(x)$. We can see that x^* is the amount you should consume to give yourself the highest utility level. At x^*, $U(x)$ is at its highest possible level. Note that at x^*, the slope of the utility function is 0 (it is flat). Mathematically, we would say

$$U'(x^*) = 0.$$

This observation gives a method of solving the utility maximization problem.

1. Take the derivative of the utility function (called the first order condition)
2. Set it equal to zero
3. Solve for the value of x that makes the statement hold.

Example 15 $U = 16x - x^2$

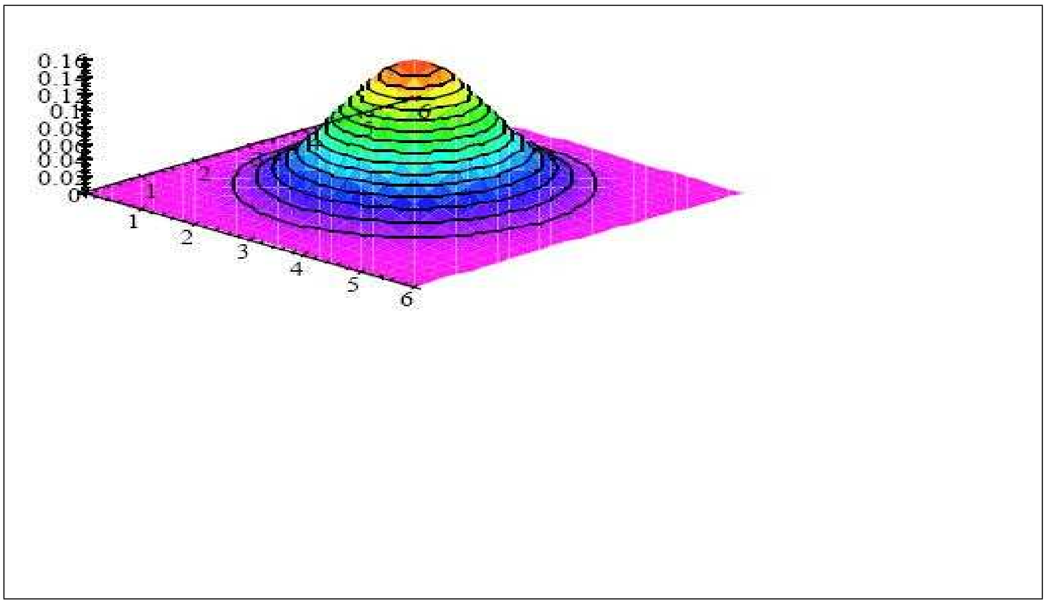


Lets solve this problem analytically. Then we can plot it and visually see for ourselves that we have found a maximum.

1. Find the first order condition:
2. Set the first order condition equal to zero
3. Solve for x.

What does x^* represent?
What is the sign of the marginal utility? What does it tell us?

Example 16 $U = 10 - x^2$



1.5.2 Multivariate function

In fact, we know that we are typically thinking of consuming mulitple goods, so we need to think about a multivariate function. Again, for the moment assume we have no budget constraint.

Graphically, we now want to find the values of x_1 and x_2 that give us the highest utility level.

Now our problem is three dimensional rather than two dimensional. Parallel to the univariate problem, we can see that the maximum occurs where the slope of function with respect to x_1 is zero AND the slope of the function with respect to x_2 is zero. Notice that if just one of these conditions were satisified, it would not be enough to guarantee that we were at the maximum. This is equivalent to saying that we need to find the values of x_1 and x_2 such that both first order conditions are equal to zero. At this point, utility will be maximized. Mathematically, utility will be maximized when

$$\begin{aligned} U_1\left(x_1^*, x_2^*\right) &= 0 \\ &\text{and} \\ U_2\left(x_1^*, x_2^*\right) &= 0. \end{aligned}$$

This observation gives a method of solving the utility maximization problem.

1. Take the derivative of the utility function with respect to x_1 and set it equal to zero
2. Take the derivative of the utility function with respect to x_2 and set it equal to zero
3. Solve the equations for the value of x_1 and x_2 that makes the statement hold.

2 Multivariate Optimization Continued

So far, to understand the process, we have examined optimization in the case of no budget constraint. This is unrealistic. So now we want to adjust our model to determine the optimal consumption levels given our budget constraint. We want to choose x_1 and x_2 to maximize our utility given our budget constraint. In math notation, this is:

$$\underset{x_1,x_2}{Max} U(x_1, x_2) \ s.t. \ p_1x_1 + p_2x_2 = m$$

We have a function that we are trying to maximize and a budget equation that limits the values of x_1 and x_2. Recall that the budget equation can be written as

$$x_2 = \frac{m - p_1x_1}{p_2}.$$

In other words, x_2 can be expressed as a function of x_1. We can substitute in this equation for x_2 in the function that we are trying to maximize so that our new problem is just a function of one variable, x_1 :

$$\underset{x_1}{Max} U(x_1, \frac{m - p_1x_1}{p_2})$$

Now this is just like our old univariate problem.

Example 17 \$6 *to spend on soup* (S) *and salad* (V). *1 ounce of soup costs 0.25 and 1 ounce of salad costs \$0.50.*

$$U(S, V) = \frac{1}{2}\ln S + \frac{1}{2}\ln V$$

How much of each should you buy?

$$\begin{aligned} \underset{V,S}{Max} \frac{1}{2}\ln S + \frac{1}{2}\ln V \ s.t. \frac{1}{4}S + \frac{1}{2}V &= 6 \\ \underset{V,S}{Max} \frac{1}{2}\ln S + \frac{1}{2}\ln V \ s.t. \ S &= 24 - 2V \\ & \underset{S}{Max} \frac{1}{2}\ln(24 - 2V) + \frac{1}{2}\ln V \end{aligned}$$

The first-order condition is:

$$\begin{aligned} \frac{1(-2)}{2(24 - 2V)} + \frac{1}{2V} &= 0 \\ 24 - 2V &= 2V \\ 4V &= 24 \\ V &= 6 \end{aligned}$$

Substituting back into budget constraint,

$$\begin{aligned} S &= 24 - 2V \\ S &= 24 - 2(6) \\ S &= 12 \end{aligned}$$

So the optimal amount of soup is 12 ounces and the optimal amount of salad is 6 ounces.

2.1 General result

Let's show that our general utility maximization result,

$$MRS = -\frac{p_1}{p_2},$$

is the solution to the maximization problem:

$$\begin{aligned} \underset{x_1,x_2}{Max} U(x_1,x_2) \ s.t. \ p_1x_1 + p_2x_2 &= m \\ & \underset{x_1}{Max} U(x_1, \frac{m - p_1x_1}{p_2}) \end{aligned}$$

Taking the FOC with respect to x_1, we get

$$\begin{aligned} U_1 &= MU_1 - \frac{p_1}{p_2} MU_2 = 0 \\ \frac{MU_1}{MU_2} &= \frac{p_1}{p_2} \end{aligned}$$

This gives us the familiar interpretation of the optimization result: you will be maximizing your utility when you will get an equal increase in utility from spending \$1 on either x_1 or x_2. Equivalently,

$$\begin{aligned} -\frac{MU_1}{MU_2} &= -\frac{p_1}{p_2} \\ MRS &= -\frac{p_1}{p_2} \end{aligned}$$

This is our familiar tangency condition: when you maximize your utility subject to a budget constraint, the slope of the indifference curve will be equal to the slope of the budget line.

Example 18 *Find optimal bundle for* $U = \frac{1}{4}\ln x_1 + \frac{3}{4}\ln x_2$, *when* $p_1 = 2, p_2 = 1, m = 16$

What type of utility? Cobb-Douglas

$$\underset{x_1,x_2}{Max} \frac{1}{4}\ln x_1 + \frac{3}{4}\ln x_2 \quad s.t. \ 2x_1 + x_2 = 16$$

Note: 2 variables, 2 equations — substitute
Step 1: Solve budget constraint for x_2 (or x_1)

$$x_2 = 16 - 2x_1$$

Step 2: Substitute into utility function. Now utility is just function of 1 variable.

$$\underset{x_1}{Max} \frac{1}{4}\ln x_1 + \frac{3}{4}\ln(16 - 2x_1)$$

Remember your calculus. This problem says to find the value of x_1 that is associated with finding the highest utility value. Anytime you want to maximize a function, you take the derviative, set equal to zero, and solve.

Step 3: Take derivative and set equal to zero.

$$\frac{1}{4x_1} + \frac{3(-2)}{4(16 - 2x_1)} = 0$$

Step 4: solve for x_2

$$\begin{aligned} \frac{1}{4x_1} &= \frac{6}{4(16 - 2x_1)} \\ 6x_1 &= 16 - 2x_1 \\ 8x_1 &= 16 \\ x_1 &= 2 \end{aligned}$$

Step 5: Solve for x_2, using budget relationship

$$\begin{aligned} x_2 &= 16 - 2x_1 \\ &= 16 - 2(2) \\ &= 12 \end{aligned}$$

We solved this problem for specific values of p_1, p_2, m. If any of these values changed, we would have to recalculate. But we could solve more generally for the optimal consumption levels as a function of prices and income.

$$\begin{aligned}
\underset{x_1,x_2}{Max} \alpha \ln x_1 + (1-\alpha)\ln x_2 \;\; s.t.\; p_1x_1 + p_2x_2 &= m \\
\underset{x_1,x_2}{Max} \alpha \ln x_1 + (1-\alpha)\ln x_2 \;\; s.t.\; x_2 &= \frac{m - p_1x_1}{p_2} \\
& \underset{x_1}{Max} \alpha \ln x_1 + (1-\alpha)\ln\left(\frac{m}{p_2} - \frac{p_1}{p_2}x_1\right) \\
\frac{\alpha}{x_1} + \frac{(1-\alpha)\left(-\frac{p_1}{p_2}\right)}{\left(\frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)} &= 0 \\
\frac{\alpha}{x_1} &= \frac{(1-\alpha)\left(\frac{p_1}{p_2}\right)}{\left(\frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)} \\
x_1(1-\alpha)\left(\frac{p_1}{p_2}\right) &= \alpha\left(\frac{m}{p_2} - \frac{p_1}{p_2}x_1\right) \\
x_1\frac{p_1}{p_2}[1-\alpha+\alpha] &= \alpha\frac{m}{p_2} \\
x_1\frac{p_1}{p_2} &= \alpha\frac{m}{p_2} \\
x_1 &= \alpha\frac{m}{p_2}\frac{p_2}{p_1} \\
x_1 &= \alpha\frac{m}{p_1}
\end{aligned}$$

Substituting back into the budget equation:

$$\begin{aligned}
x_2 &= \frac{m}{p_2} - \frac{p_1}{p_2}\left(\alpha\frac{m}{p_1}\right) \\
&= \frac{m}{p_2} - \alpha\frac{m}{p_2} \\
&= \frac{m}{p_2}(1-\alpha)
\end{aligned}$$

Using the values of m, p_1, p_2, α from before,

$$\begin{aligned}
x_1 &= \frac{1}{4}\frac{16}{2} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
x_2 &= \frac{16}{1}\left(\frac{3}{4}\right) \\
&= 12
\end{aligned}$$

These are the same answers as before.

2.1.1 Application: Commuting Utility

Goal: examine preferences over commuting.

Want to write utility function that includes those attributes that we think might affect the commuting decision.

$$U = \beta_1 TW + \beta_2 TT + \beta_3 C$$

Goal is to estimate these preference parameters. How important is walking time versus total trip time versus cost.

One study examined this issue and made the following estimates

$$U = -.147TW + -.0411TT - 2.24C$$

Want to interpret what these mean:

- Made worse off by increases in walking time, increases in total travel time, and increases in cost.
- Most important factors are: cost, then walking time, then total travel time
- $$MRS(x_1, x_2) = MRS(TW, TT) = -\frac{MU_{TT}}{MU_{TW}} = -\frac{-.147}{-.0411} \simeq -3$$

  People are willing to spend 3 more minutes traveling in order to decrease traveling time by 1 minute
- $$MRS(TT, C) = -\frac{MU_{TT}}{MU_C} = -\frac{-.0411}{-2.24} = -.0183$$

  Each minute spend traveling is approximately worth 2 cents to people. Therefore, individual value an hour of traveling at \$1.10. They would be willing to spend \$1.10 for each hourly decrease in commuting time.

Can use these estimates to answer questions such as : should we increase bus service? To answer this question, need to compare costs and benefits. Can estimate benefits from studies such as this.